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JUSTIFICATION OF THE FAST MULTIPOLE METHOD FOR THE STOKES SYSTEM. PART II. EXTERIOR DOMAIN PROBLEMS

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We consider the exterior domain problems of Dirichlet and Neumann type of the two-dimensional Stokes equations. For the solution of this boundary value problem we choose a potential ansatz and show that for the reduction of the computational costs, the fast multipole method of Greengard and Rokhlin can be used. Therefore, we find a complex representation of the hydrodynamical potentials and provide statements about the corresponding multipole and Taylor expansions, as well as the appropriate translation, rotation and conversion operators. The theoretical statements are illustrated by numerical experiments. Bibliography: 15 *titles.*

1 Introduction

For the solution of the stationary linearized Navier–Stokes system (so-called stationary Stokes equations' system), different integral equation formulations are known. In particular, the extensive theory based on complex variables is developed in [1, 2]. Khoromskij et al. [3] presented a technique of almost linear complexity for solving elliptical partial differential equations based on their reduction to the interface. In the case of the Stokes equations, this interface reduction method is based on either the stream function-vorticity formulation or the use of the special Poincaré–Steklov operator. An alternative approach, using the fundamental solution and hydrodynamical potentials, can be found in [4]–[6] and provides the analytical foundation for our discussion.

We restrict our attention to the classical boundary value problems for the Stokes equations in exterior domains. By the methods presented in [4, 7], we represent the solution in form of hydrodynamical potentials, whose unknown densities are solutions of uniquely solvable boundary integral equations. For the numerical solution of these integral equations with m points in the discretization of the boundary, a dense, nonsymmetrical linear system have to be solved. The direct inversion of such a system requires $O(m^3)$ operations. The most work intensive part by the application of the iterative methods is the computation of matrix-vector products. A

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direct calculation of a matrix-vector product requires $O(m^2)$ operations. Since one uses the fast multipole method (FMM) [8]–[10] to compute matrix-vector products, the computational costs can be reduced up to $O(m \log^\alpha m)$. Furthermore, an iterative method with FMM fast multipole method does not require storage of a dense matrix [10]–[12].

In this paper, we obtain a new representation of hydrodynamical tensors based on complex variables, verify the corresponding multipole and Taylor expansions and provide theorems about the translation and conversion of these expansions. All these analytical preliminaries allow us to use the fast multipole method for the numerical solution of our boundary integral equations and for the computation of the approximate hydrodynamical potentials. For m nodes in the discretization of the boundary, the velocity part of hydrodynamical potentials consists of m^2 symmetrical 2×2-blocks with elements having three different prescription. Therefore, we need several FMM cycles to calculate one matrix-vector product in our case. As in the case of the interior Dirichlet problem presented in [13], this method essentially reduces the computational costs, which will be demonstrated by numerical test computations.

In the following section, we briefly review the analytical foundation concerning the solvability of the boundary value problem for the Stokes equations in exterior domains and present the corresponding integral equation formulations. In Section 3, we deduce representations of hydrodynamical single layer potential tensor and of its normal stress based on complex variables. Section 4 contains a detail description of the analytical preliminaries, which are necessary for the application of the fast multipole method. The performance of the proposed method is illustrated with numerical examples in Section 5.

2 Solvability of the Exterior Boundary Value Problems of Neumann and Dirichlet Type for the System of Stokes Equations

We consider the two-dimensional Stokes system

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in} \quad G^*, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad G^* \end{aligned} \tag{2.1}$$

with boundary conditions of two types:

$$\text{Neumann:} \qquad T\mathbf{n} = \mathbf{b} \quad \text{on} \quad \partial G^*, \tag{2.2}$$

$$\text{Dirichlet:} \qquad \mathbf{u} = \mathbf{b} \quad \text{on} \quad \partial G^*. \tag{2.3}$$

The domain $G^* \subset \mathbb{R}^2$ with the boundary $\partial G^* \in \mathbb{C}^2$, where the system (2.1) is to be solved, is exterior to a simple closed curve. Here $\mathbf{u}$ is the velocity vector, p is the (scalar) pressure of a viscous incompressible fluid with conservative external forces, $\mathbf{n}$ denotes the outward unit normal vector on ∂G^*, and T is the stress tensor defined by

$$T_{ij} := -p\,\delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad i, j = 1, 2 \tag{2.4}$$

where δ_{ij} is the Kronecker delta. Physically, the Neumann condition corresponds to the prescribing force distribution on the boundary. The functions $\mathbf{u}$ and $\mathbf{b}$ are two-dimensional vector-valued functions in G^* and on ∂G^* respectively, Δ denotes the Laplacian, ∇ is the gradient, and $\nabla\cdot$ is the divergence in $\mathbb{R}^2$. We assume that the boundary value $\mathbf{b}$ is given continuously on ∂G^*.

In the following two lemmas, we outline the main results for the solvability of the Stokes system in the cases (2.2) and (2.3) without detailed investigation of the corresponding problems (cf. details in [5]–[7]).

Lemma 2.1. *For any continuous vector-valued function* $\mathbf{b}$ *such that*

$$\int\limits_{\partial G^*} \mathbf{b}\,\mathrm{do} = 0 \tag{2.5}$$

the exterior Neumann problem for the Stokes equations

$$\begin{aligned} -\Delta\mathbf{u} + \nabla p &= \mathbf{0} \quad \textit{in} \ \ G^*, \\ \nabla\cdot\mathbf{u} &= 0 \quad \textit{in} \ \ G^*, \\ T\mathbf{n} &= \mathbf{b} \quad \textit{on} \ \ \partial G^* \end{aligned} \tag{2.6}$$

has a solution $(\mathbf{u}, p)$ *which satisfies the decay conditions*

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &= \mathscr{O}(1) \quad \textit{as} \ \ |\mathbf{x}| \longrightarrow \infty, \\ \nabla\mathbf{u}(\mathbf{x}) &= \mathscr{O}(|\mathbf{x}|^{-2}) \quad \textit{as} \ \ |\mathbf{x}| \longrightarrow \infty, \\ p(\mathbf{x}) &= \mathscr{O}(|\mathbf{x}|^{-2}) \quad \textit{as} \ \ |\mathbf{x}| \longrightarrow \infty. \end{aligned} \tag{2.7}$$

This solution is unique to within a constant vector $\mathbf{c}$ *and if* $\mathbf{u} \longrightarrow 0$ *as* $|\mathbf{x}| \longrightarrow \infty$*, then the solution of the exterior Neumann problem for the Stokes equations is unique.*

If we look for a solution of the problem (2.6) as in [6] in the form

$$\begin{aligned} \mathbf{u} &= \nabla \ln|\mathbf{x}| \int\limits_{\partial G} \mathbf{q}(\mathbf{y})\mathbf{n}(\mathbf{y})\mathrm{do} + E\mathbf{q}(\mathbf{x}), \\ p &= \frac{1}{2\pi}\int\limits_{\partial G} \frac{\mathbf{x}-\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\mathbf{q}(\mathbf{y})\mathrm{do}, \quad \mathbf{x} \in G^*, \end{aligned} \tag{2.8}$$

where

$$E\mathbf{q}(\mathbf{x}) := \int\limits_{\partial G} \widetilde{E}(\mathbf{x}-\mathbf{y})\mathbf{q}(\mathbf{y})\mathrm{ds}, \quad \mathbf{x} \in G^*, \tag{2.9}$$

is the velocity part of the hydrodynamical single layer potential with the corresponding 2×2-matrix

$$\widetilde{E}_{ij}(\mathbf{x}-\mathbf{y}) = \frac{1}{4\pi}\frac{(x_i-y_i)(x_j-y_j)}{|\mathbf{x}-\mathbf{y}|^2} - \delta_{ij}\ln|\mathbf{x}-\mathbf{y}|, \quad i,j=1,2, \tag{2.10}$$

then the unknown density $\mathbf{q}$ can be obtained using well known continuity and jump relations of the hydrodynamical potentials (see, e.g., [5]) from the following uniquely solvable (cf. [6]) system of boundary integral equations:

$$-\frac{1}{2}\mathbf{q}(\mathbf{x}) - H\mathbf{q}(\mathbf{x}) + \left(\int\limits_{\partial G} \mathbf{q}(\mathbf{y})\cdot\mathbf{n}(\mathbf{y})\mathrm{do}_{\mathbf{y}}\right)\mathbf{g}(\mathbf{x}) = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \partial G^*, \tag{2.11}$$

where

$$H\mathbf{q}(\mathbf{x}) := \int\limits_{\partial G} \widetilde{H}(\mathbf{x},\mathbf{y})\mathbf{q}(\mathbf{y})\mathrm{d}o_{\mathbf{y}} \tag{2.12}$$

with

$$\widetilde{H}_{ij}(\mathbf{x},\mathbf{y}) = T_{ij}(\widetilde{E}(\mathbf{x}-\mathbf{y}))\mathbf{n}(\mathbf{x}) = \frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|^4}, \quad i,j=1,2, \tag{2.13}$$

and

$$g_i(\mathbf{x}) := \{T_{ij}(\nabla \ln|\mathbf{x}|)\, n_j\}|_{\partial G} = 2\, n_j \frac{\partial^2}{\partial x_i \partial x_j} \ln|\mathbf{x}|, \quad i,j=1,2.$$

For the numerical solution of (2.11), one can parameterize these equations with $\gamma : [0,1] \longrightarrow \partial G$ and discretize them using, for example, the Nyström method, by m discretization points. Setting $h := 1/m$, for (2.11) one obtains the following system of linear equations, which can be solved, for example, by the iterative method:

$$-\frac{1}{2}\widetilde{\mathbf{q}}(ih) - h\sum_{i\neq j=1}^{m} \widetilde{H_\gamma}(ih,jh)\widetilde{\mathbf{q}}(jh) - h\mathscr{K}\widetilde{\mathbf{q}}(jh) + h\widetilde{\mathbf{g}}(ih)\sum_{j=1}^{m}\widetilde{\mathbf{f}}(jh)\cdot\widetilde{\mathbf{n}_\gamma}(jh) = \widetilde{\mathbf{b}}(ih), \tag{2.14}$$

with $i,j = 1,\dots,m$, where

$$\begin{aligned} &\widetilde{\mathbf{q}}(ih) := \mathbf{q}(\gamma(ih)), \quad \widetilde{\mathbf{b}}(ih) := \mathbf{b}(\gamma(ih)), \quad \widetilde{\mathbf{g}}(ih) := \mathbf{g}(\gamma(ih)), \\ &\widetilde{\mathbf{n}_\gamma}(jh) := \mathbf{n}(\gamma(jh))|\dot{\gamma}(jh)|, \quad \widetilde{H_\gamma}(ih,jh) := \widetilde{H}(ih,jh)\mid\dot{\gamma}(jh)|, \end{aligned} \tag{2.15}$$

and a curvature matrix $\mathscr{K}$, which can be explicitly specified for a smooth boundary.

Lemma 2.2. *The exterior Dirichlet problem for the Stokes equations*

$$\begin{aligned} &-\Delta\mathbf{u} + \nabla p = \mathbf{0} \quad \text{in} \ \ G^*, \\ &\nabla\cdot\mathbf{u} = 0 \quad \text{in} \ \ G^*, \\ &\mathbf{u} = \mathbf{b} \quad \text{on} \ \ \partial G^* \end{aligned} \tag{2.16}$$

has a unique solution $(\mathbf{u}, p)$ *in the class of functions possessing the property* (2.7).

As in [7], we choose a potential ansatz for the solution of (2.16) and the following representation for the velocity part of the solution:

$$\mathbf{u}(\mathbf{x}) := D\mathbf{q}(\mathbf{x}) - \eta EM\mathbf{q}(\mathbf{x}) - \alpha\int\limits_{\partial G^*}\mathbf{q}\mathrm{d}o, \quad 0<\eta\in\mathbb{R}, \quad 0\neq\alpha\in\mathbb{R}, \tag{2.17}$$

where

$$D\,\mathbf{q}\,(\mathbf{x}) := \int\limits_{\partial G}\widetilde{D}(\mathbf{x},\mathbf{y})\mathbf{q}(\mathbf{y})\mathrm{d}o_{\mathbf{y}} \tag{2.18}$$

denotes the velocity part of the hydrodynamical double layer potential with the corresponding 2×2-matrix

$$\widetilde{D}_{ij}(\mathbf{x},\mathbf{y}) := -\frac{1}{\pi}\frac{(x_i-y_i)(x_j-y_j)(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^4}, \quad i,j=1,2, \tag{2.19}$$

$EM\mathbf{q}(\mathbf{x})$ is the velocity part of the hydrodynamical single layer potential, which is applied to the image of the projection operator $M : C(\partial G^*)^2 \longrightarrow C(\partial G^*)^2$ defined by

$$\mathbf{q} \longrightarrow M\mathbf{q} := \mathbf{q} - \mathbf{q}_M$$

with the surface mean value

$$\mathbf{q}_M := \frac{1}{|\partial G^*|} \int\limits_{\partial G^*} \mathbf{q} \mathrm{do}, \quad |\partial G^*| = \int\limits_{\partial G^*} 1 \, \mathrm{do}.$$

The unknown density $\mathbf{q}$ in (2.17) can be found as a unique solution of the following system of boundary integral equations (cf. [7]):

$$-\frac{1}{2}\mathbf{q}(\mathbf{x}) + D\mathbf{q}(\mathbf{x}) - \eta EM\mathbf{q}(\mathbf{x}) - \alpha|\partial G^*|(\mathbf{I}_2 - M)\mathbf{q}(\mathbf{x}) = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \partial G^*, \tag{2.20}$$

where $0 < \eta \in \mathbb{R}$, $0 \neq \alpha \in \mathbb{R}$, and $\mathbf{I}_2$ is the 2×2-identity matrix.

The most costly part by the iterative solution of the systems (2.11) and (2.20), as well as by the approximative computation of the hydrodynamical potentials in (2.8) and (2.17), is the calculation of matrix-vector products. This procedure requires $\mathscr{O}(m^2)$ multiplications, but the computational costs can be essentially reduced if the corresponding matrix components $\mathbf{k}(\mathbf{x}^i, \mathbf{x}^j)$ can be approximated by the term with separable variables:

$$\mathbf{k}(\mathbf{x}^i, \mathbf{x}^j) \approx \sum_{l=1}^{M} \mathbf{u}_l(\mathbf{x}^i)\, \mathbf{v}_l(\mathbf{x}^j). \tag{2.21}$$

In this case, the left hand side of (2.14) is of the form

$$\left(\sum_{j=1}^{m} \mathbf{k}(\mathbf{x}^i, \mathbf{x}^j)\mathbf{f}(\mathbf{x}^j) \right)_i \approx \left(\sum_{l=1}^{M} \mathbf{u}_l(\mathbf{x}^i) \sum_{j=1}^{m} \mathbf{v}_l\,(\mathbf{x}^j)\mathbf{f}(\mathbf{x}^j) \right)_i. \tag{2.22}$$

The product (2.22) only requires $\mathscr{O}(Mm)$ floating operations, which is much faster than $\mathscr{O}(m^2)$ by increasing m. This idea is used by the fast multipole method (cf., for example, [10, 14]). Hence, if we find a representation of the form (2.21) for the matrices (2.10), (2.13), and (2.19), then we will be able to use the fast multipole method by the calculation of hydrodynamical potentials and essentially reduce the computational costs by the numerical solution of our exterior problems (2.6). We have already done it in the case of the interior Dirichlet problem for the Stokes equations in [13], where we presented a suitable representation of the velocity part of the hydrodynamical double layer tensor (2.19) and derived the corresponding multipole and Taylor expansion, as well as its shifting and converting operators. Therefore, we will only discuss in the following the hydrodynamical single layer tensor (2.10) and its normal stress tensor (2.13).

3 Complex–Valued Representation of the Hydrodynamical Potentials

Let $z, z_0, z_1, \ldots, z_m \in \mathbb{C}$ be given points in the complex plane. We will not make any distinction between a point $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ and a point $x_1 + i\,x_2 = z \in \mathbb{C}$, and set for

$\mathbf{x}, y, n \in \mathbb{R}^2$

$$\begin{aligned}
&\mathbf{x} = (x_1, x_2) \leftrightarrow z = x_1 + i x_2, \\
&\mathbf{y} = (y_1, y_2) \leftrightarrow z_0 = y_1 + i y_2, \qquad (3.1)\\
&\mathbf{n}(\mathbf{x}) = (n_1(\mathbf{x}), n_2(\mathbf{x})) \leftrightarrow N = n_1(\mathbf{x}) + i n_2(\mathbf{x}), \quad M := n_2(\mathbf{x}) + i n_1(\mathbf{x}).
\end{aligned}$$

We write $\overline{N}$ and $\overline{M}$ for the conjugate of N and M and denote by $\operatorname{Re} z$ and $\operatorname{Im} z$ the real and imaginary parts of any $z \in \mathbb{C}$ respectively. Using this notation, we find

$$\begin{aligned}
&\operatorname{Re}(z - z_0) \operatorname{Re} \frac{1}{z - z_0} = (x_1 - y_1) \operatorname{Re} \frac{(x_1 - y_1) - i(x_2 - y_2)}{|z - z_0|^2} = \frac{(x_1 - y_1)^2}{|\mathbf{x} - \mathbf{y}|^2}, \\
&\operatorname{Im}(z - z_0) \operatorname{Re} \frac{1}{z - z_0} = \frac{(x_1 - y_1)(x_2 - y_2)}{|\mathbf{x} - \mathbf{y}|^2}, \\
&\operatorname{Im}(z_0 - z) \operatorname{Im} \frac{1}{z - z_0} = \frac{(x_2 - y_2)^2}{|\mathbf{x} - \mathbf{y}|^2}, \\
&\operatorname{Re}(\log(z - z_0)) = \operatorname{Re}(\ln|z - z_0| + i(\arg(z - z_0) + 2k\pi)) = \ln|\mathbf{x} - \mathbf{y}|.
\end{aligned}$$

Then for the velocity part of the hydrodynamic single layer tensor we obtain the following complex representation:

$$\begin{aligned}
&E_{11}(\mathbf{x} - \mathbf{y}) \leftrightarrow E_{11}(z - z_0) = \frac{1}{4\pi}\left(\operatorname{Re}(z - z_0) \operatorname{Re} \frac{1}{z - z_0} - \operatorname{Re}(\log(z - z_0)) \right), \\
&E_{12}(\mathbf{x} - \mathbf{y}) \leftrightarrow E_{12}(z - z_0) = E_{21}(z - z_0) = \frac{1}{4\pi} \operatorname{Im}(z - z_0) \operatorname{Re} \frac{1}{z - z_0}, \qquad (3.2)\\
&E_{22}(\mathbf{x} - \mathbf{y}) \leftrightarrow E_{22}(z - z_0) = \frac{1}{4\pi}\left(\operatorname{Im}(z_0 - z) \operatorname{Im} \frac{1}{z - z_0} - \operatorname{Re}(\log(z - z_0)) \right).
\end{aligned}$$

To find a complex version of the normal stress tensor from (2.13), corresponding to the hydrodynamical single layer potential, we note that

$$\begin{aligned}
\operatorname{Re} \frac{N}{z - z_0} &= \frac{|z - z_0|^2}{|z - z_0|^2} \operatorname{Re} \frac{(n_1 + i n_2)((x_1 - y_1) - i(x_2 - y_2))}{|z - z_0|^2} \\
&= \frac{n_1 (x_1 - y_1)^3 + n_1 (x_1 - y_1)(x_2 - y_2)^2}{|z - z_0|^4} + \frac{n_2 (x_1 - y_1)^2 (x_2 - y_2) + n_2 (x_2 - y_2)^3}{|z - z_0|^4},
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
\operatorname{Re} \frac{N}{(z - z_0)^2} &= \operatorname{Re} \frac{(n_1 + i n_2)((x_1 - y_1) - i(x_2 - y_2))^2}{|z - z_0|^4} \\
&= \frac{n_1 (x_1 - y_1)^2 - n_1 (x_2 - y_2)^2 + 2 n_2 (x_1 - y_1)(x_2 - y_2)}{|z - z_0|^4},
\end{aligned} \tag{3.4}$$

and

$$\operatorname{Re} \frac{\overline{N}}{(z - z_0)^2} = \frac{n_1 (x_1 - y_1)^2 - n_1 (x_2 - y_2)^2 - 2 n_2 (x_1 - y_1)(x_2 - y_2)}{|z - z_0|^4}, \tag{3.5}$$

as well as

$$\frac{1}{4} \operatorname{Re}(z - z_0)\left(\operatorname{Re} \frac{N}{(z - z_0)^2} - \operatorname{Re} \frac{\overline{N}}{(z - z_0)^2} \right) = \frac{n_2 (x_1 - y_1)^2 (x_2 - y_2)}{|z - z_0|^4} \tag{3.6}$$

and

$$\frac{1}{4}\,\mathrm{Im}\,(z-z_0)\left(\,\mathrm{Re}\,\frac{M}{(z-z_0)^2}-\mathrm{Re}\,\frac{\overline{M}}{(z-z_0)^2}\right)=\frac{n_1\,(x_1-y_1)(x_2-y_2)^2}{|z-z_0|^4}. \tag{3.7}$$

Using (3.3)–(3.7), we obtain the following complex representation of (2.13):

$$\begin{aligned}
\widetilde{H}_{11}(\mathbf{x},\mathbf{y}) &\leftrightarrow \widetilde{H}_{11}(z-z_0)\\
&=\frac{1}{4\,\pi}\left(\,\mathrm{Re}\,\frac{4\,N}{z-z_0}-\mathrm{Re}\,(z-z_0)\,\mathrm{Re}\,\frac{N-\overline{N}}{(z-z_0)^2}+\mathrm{Im}\,(z-z_0)\,\mathrm{Re}\,\frac{3\overline{M}+M}{(z-z_0)^2}\right),\\
\widetilde{H}_{12}(\mathbf{x},\mathbf{y}) &\leftrightarrow \widetilde{H}_{12}(z-z_0)\\
&=\widetilde{H}_{21}(z-z_0)=\frac{1}{4\,\pi}\left(\,\mathrm{Re}\,(z-z_0)\,\mathrm{Re}\,\frac{M-\overline{M}}{(z-z_0)^2}+\mathrm{Im}\,(z-z_0)\,\mathrm{Re}\,\frac{N-\overline{N}}{(z-z_0)^2}\right),\\
\widetilde{H}_{22}(\mathbf{x},\mathbf{y}) &\leftrightarrow \widetilde{H}_{22}(z-z_0)\\
&=\frac{1}{4\pi}\left(\,\mathrm{Re}\,\frac{4\,N}{z-z_0}-\mathrm{Re}\,(z-z_0)\,\mathrm{Re}\,\frac{3N+\overline{N}}{(z-z_0)^2}-\mathrm{Im}\,(z-z_0)\,\mathrm{Re}\,\frac{M-\overline{M}}{(z-z_0)^2}\right).
\end{aligned} \tag{3.8}$$

A complex representation of the double layer tensor (2.19) is similar to the representation (3.8) of the normal stress tensor and was already derived and discussed in [13].

4 Analytical Investigation of the Hydrodynamical Tensors

The iterative solution of the linear system (2.14) requires the evaluation of the finite sums of the form

$$\mathrm{Re}\,\left(N\sum_{j=1}^{m}\frac{q_j}{z-z_j}\right)\qquad\text{and}\qquad\sum_{j=1}^{m}q_j\,\mathrm{Re}\,(z-z_j)\,\mathrm{Re}\,\frac{N-\overline{N}}{(z-z_j)^2}, \tag{4.1}$$

giving representations (3.2) and (3.8). By the calculation of the solution of the exterior Neumann problem giving by (2.8), we have to find sums of the types

$$\sum_{j=1}^{m}q_j\,\mathrm{Re}\,(z-z_j)\,\mathrm{Re}\,\frac{1}{z-z_j}\qquad\text{and}\qquad\sum_{j=1}^{m}q_j\,\mathrm{Re}\,(\log(z-z_j)). \tag{4.2}$$

In order to speed up the evaluation of the sums from (4.1) and (4.2), we would like to use the fast multipole method and, therefore, have to find for them the corresponding multipole and Taylor expansions, as well as appropriate translation and conversion operators. Such investigations for the terms

$$\sum_{j=1}^{m}q_j\,\mathrm{Re}\,\frac{N_j}{z-z_j}\qquad\text{and}\qquad\sum_{j=1}^{m}q_j\,\mathrm{Re}\,(\log(z-z_j)),$$

where N_j is the complex representation of $\mathbf{n}(\mathbf{y})$, can be found, for example, in [14], where they were presented in the context of the Laplace equation. Thus, we only discuss in detail the terms

$$\begin{aligned}
&\operatorname{Re}\left(N\sum_{j=1}^{m}\frac{q_j}{z-z_j}\right),\\
&\sum_{j=1}^{m} q_j \operatorname{Re}(z-z_j)\operatorname{Re}\frac{1}{z-z_j},\\
&\sum_{j=1}^{m} q_j \operatorname{Re}(z-z_j)\ \operatorname{Re}\frac{N-\overline{N}}{(z-z_j)^2}.
\end{aligned} \tag{4.3}$$

To obtain the multipole expansion for the terms in (4.3), we need the following assertion.

Lemma 4.1. *Let $q_0 \in \mathbb{R}$ and $N, z_0 \in \mathbb{C}$ be given. Then for any z with $|z| > |z_0|$ we have*

(i) $\Phi^H_{z_0,1}(z) := \operatorname{Re}\left(\dfrac{q_0\,N}{z-z_0}\right) = \operatorname{Re}\left(q_0\,N\displaystyle\sum_{k=0}^{\infty}\frac{z_0^k}{z^{k+1}}\right),$

(ii) $\Phi^E_{z_0,1}(z) := q_0\operatorname{Re}(z-z_0)\operatorname{Re}\dfrac{1}{z-z_0} = q_0\operatorname{Re} z\operatorname{Re}\displaystyle\sum_{k=0}^{\infty}\frac{z_0^k}{z^{k+1}} - q_0\operatorname{Re}\sum_{k=0}^{\infty}\frac{z_0^k\operatorname{Re} z_0}{z^{k+1}}$

(iii) $$\begin{aligned}\Phi^H_{z_0,2}(z) &:= q_0\operatorname{Re}(z-z_0)\operatorname{Re}\frac{N-\overline{N}}{(z-z_0)^2}\\ &= q_0\operatorname{Re} z\operatorname{Re}\left((N-\overline{N})\sum_{k=0}^{\infty}(k+1)\frac{z_0^k}{z^{k+2}}\right) - q_0\operatorname{Re}\left((N-\overline{N})\sum_{k=0}^{\infty}(k+1)\frac{z_0^k\operatorname{Re} z_0}{z^{k+2}}\right)\end{aligned}$$

(iv) $\Phi^E_{z_0,2}(z) := \operatorname{Re}(q_0\log(z-z_0)) = \operatorname{Re}\left(q_0\left(\log z - \displaystyle\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{z_0}{z}\right)^k\right)\right).$

Proof. Since $|z| > |z_0|$, one obtains (i) and (ii) by elementary calculations from the expansion

$$\frac{1}{z-z_0} = \frac{1}{z}\left(1+\frac{z_0}{z}+\ldots+\left(\frac{z_0}{z}\right)^n+\ldots\right) = \sum_{k=0}^{\infty}\frac{z_0^k}{z^{k+1}},$$

and (iii) from

$$\frac{1}{(z-z_0)^2} = \frac{1}{z^2}\left(1+\frac{z_0}{z}+\ldots+\left(\frac{z_0}{z}\right)^n+\ldots\right)^2 = \sum_{k=0}^{\infty}(k+1)\frac{z_0^k}{z^{k+2}}\,.$$

To show the forth statement of the lemma, we note that

$$\log(z-z_0) - \log z = \log(1-\frac{z_0}{z})$$

for $|z_0/z| < 1$. Thus, (iv) follows from the expansion

$$\log\left(1-\frac{z_0}{z}\right) = (-1)\sum_{k=1}^{\infty}\frac{z_0^k}{k\,z^k}. \qquad \square$$

For the further investigations, we introduce the following notation:

$$\Phi_{H_1}(z) := \sum_{j=1}^{m} \Phi^{H}_{z_j,1}(z) = \operatorname{Re}\left(N \sum_{j=1}^{m} \frac{q_j}{z - z_j}\right), \tag{4.4}$$

$$\Phi_{E_1}(z) := \sum_{j=1}^{m} \Phi^{E}_{z_j,1}(z) = \sum_{j=1}^{m} q_j \operatorname{Re}(z - z_j) \operatorname{Re} \frac{1}{z - z_j}, \tag{4.5}$$

$$\Phi_{H_2}(z) := \sum_{j=1}^{m} \Phi^{H}_{z_j,2}(z) = \sum_{j=1}^{m} q_j \operatorname{Re}(z - z_j) \operatorname{Re} \frac{N - \overline{N}}{z - z_j}, \tag{4.6}$$

$$\Phi_{E_2}(z) := \sum_{j=1}^{m} \Phi^{E}_{z_j,2}(z) = \sum_{j=1}^{m} q_j \operatorname{Re}(\log(z - z_j)). \tag{4.7}$$

Theorem 4.2 (multipole expansion)**.** *Let $m \in \mathbb{N}$, $q_j, r \in \mathbb{R}$, and $N, z_j \in \mathbb{C}$, $j = 1, \ldots, m$, be given. Then for any $z \in \mathbb{C}$ with $|z - z_j| > r$ the function $\Phi_{H_1}(z)$ is given by*

$$\Phi_{H_1}(z) = \operatorname{Re}\left(N \sum_{k=0}^{\infty} \frac{\widetilde{a}_k}{z^{k+1}}\right) \quad \text{with} \quad \widetilde{a}_k := \sum_{j=1}^{m} q_j z_j^k. \tag{4.8}$$

It holds for any $p \in \mathbb{N}$

$$\left| \sum_{k=p+1}^{\infty} \frac{\widetilde{a}_k}{z^k} \right| \leqslant \frac{A}{|z| - r} \left|\frac{r}{z}\right|^{p+1} \quad \text{with} \quad A := \sum_{j=1}^{m} |q_j|. \tag{4.9}$$

Furthermore,

$$\Phi_{H_2}(z) = \operatorname{Re} z \operatorname{Re}\left((N - \overline{N}) \sum_{k=1}^{\infty} \frac{a_k}{z^{k+1}}\right) - \operatorname{Re}\left((N - \overline{N}) \sum_{k=1}^{\infty} \frac{a'_k}{z^{k+1}}\right) \tag{4.10}$$

with

$$a_k := k \sum_{j=1}^{m} q_j z_j^{k-1} \quad \text{and} \quad a'_k := k \sum_{j=1}^{m} q_j z_j^{k-1} \operatorname{Re} z_j. \tag{4.11}$$

For any $p \in \mathbb{N}$ we obtain the estimates

$$\left| \sum_{k=p+1}^{\infty} \frac{a_k}{z^{k+1}} \right| \leqslant \alpha_1 \left|\frac{r}{z}\right|^{p} \quad \text{and} \quad \left| \sum_{k=p+1}^{\infty} \frac{a'_k}{z^{k+1}} \right| \leqslant \alpha_2 \left|\frac{r}{z}\right|^{p} \tag{4.12}$$

with

$$\alpha_i := \frac{A_i(r + (p+1)(|z| - r))}{|z|\,(|z| - r)^2}, \quad i = 1, 2, \quad A_1 := \sum_{i=1}^{m} |q_j|, \qquad A_2 := \sum_{i=1}^{m} |q_j \operatorname{Re} z_j|. \tag{4.13}$$

Proof. One obtains the multipole expansion (4.8) from the definition of $\Phi_{H_1}(z)$ given in (4.4) and Lemma 4.1 (i). To prove the estimate in (4.9), we substitute for $\widetilde{a}_k$ the corresponding expression in (4.8) and obtain

$$\left| \sum_{k=p+1}^{\infty} \frac{\widetilde{a}_k}{z^k} \right| \leqslant A \sum_{k=p+1}^{\infty} \frac{r^k}{|z|^{k+1}} \leqslant \frac{A}{|z|} \left|\frac{r}{z}\right|^{p+1} \sum_{k=0}^{\infty} \left|\frac{r}{z}\right|^{k} = \frac{A}{|z| - r} \left|\frac{r}{z}\right|^{p+1}.$$

For the multipole expansion of $\Phi_{H_2}(z)$ we find with Lemma 4.1 (iii)

$$\Phi_{H_1}(z) = \sum_{j=1}^{m} \mathrm{Re}(z)\,\mathrm{Re}\left((N-\overline{N})\sum_{j=1}^{m}\frac{q_j}{(z-z_j)^2}\right) - \sum_{j=1}^{m}\mathrm{Re}\,(z_j)\,\mathrm{Re}\,\frac{q_j(N-\overline{N})}{(z-z_j)^2} =: T_1(z) - T_2(z),$$

where

$$T_1(z) = \mathrm{Re}\;z\,\mathrm{Re}\left((N-\overline{N})\sum_{j=1}^{m}\frac{q_j}{z^2}\sum_{k=1}^{\infty}k\frac{z_j^{k-1}}{z^{k-1}}\right) = \mathrm{Re}\;z\,\mathrm{Re}\left((N-\overline{N})\sum_{k=1}^{\infty}\frac{a_k}{z^{k+1}}\right)$$

with

$$a_k = k\sum_{j=1}^{m} q_j z_j^{k-1}$$

and

$$T_2(z) = \sum_{j=1}^{m}\mathrm{Re}\;\left((N-\overline{N})\frac{q_j\,\mathrm{Re}\;z_j}{(z-z_j)^2}\right) = \mathrm{Re}\;\left((N-\overline{N})\sum_{k=1}^{\infty}\frac{a_k'}{z^{k+1}}\right)$$

with

$$a_k' = k\sum_{j=1}^{m} q_j z_j^{k-1}\,\mathrm{Re}\;z_j.$$

Further, we obtain the expressions in (4.12) and (4.13) by

$$\left|\sum_{k=p+1}^{\infty}\frac{a_k}{z^{k+1}}\right| \leqslant \frac{A_1}{|z|^2}\left|\frac{r}{z}\right|^p\sum_{k=0}^{\infty}(k+p+1)\left|\frac{r}{z}\right|^k = \alpha_1\left|\frac{r}{z}\right|^p$$

with

$$\alpha_1 = \frac{A_1(r+(p+1)(|z|-r))}{|z|\,(|z|-r)^2}.$$

□

The following theorem presents the multipole expansion for the terms from (4.2).

Theorem 4.3 (multipole expansion)**.** *Let $m\in\mathbb{N}$, $q_j, r\in\mathbb{R}$, and $N, z_j\in\mathbb{C}$, $j=1,\ldots,m$, be given. Then for any $z\in\mathbb{C}$ with $|z-z_j|>r$ the function $\Phi_{E_1}(z)$ is given by*

$$\Phi_{E_1}(z)\; = \mathrm{Re}\;z\,\mathrm{Re}\sum_{k=0}^{\infty}\frac{a_k}{z^{k+1}} - \mathrm{Re}\sum_{k=0}^{\infty}\frac{a_k'}{z^{k+1}} \tag{4.14}$$

with

$$a_k = \sum_{j=1}^{m} q_j z_j^k \quad \textit{and} \quad a_k' = \sum_{j=1}^{m} q_j z_j^k\,\mathrm{Re}\;z_j. \tag{4.15}$$

It holds for any $p\in\mathbb{N}$

$$\left|\sum_{k=p+1}^{\infty}\frac{a_k}{z^k}\right| \leqslant \frac{A_1}{|z|-r}\left|\frac{r}{z}\right|^{p+1} \quad \textit{with} \quad A_1 := \sum_{j=1}^{m}|q_j| \tag{4.16}$$

and

$$\left|\sum_{k=p+1}^{\infty}\frac{a'_k}{z^k}\right| \leqslant \frac{A_2}{|z|-r}\left|\frac{r}{z}\right|^{p+1} \quad \textit{with} \quad A_2 := \sum_{j=1}^{m}|q_j \operatorname{Re} z_j|. \tag{4.17}$$

Furthermore,

$$\Phi_{E_2}(z) = Q\log z + \sum_{k=1}^{\infty}\frac{\widetilde{a}_k}{z^k} \quad \textit{with} \quad Q = \sum_{j=1}^{m} q_j, \quad \widetilde{a}_k = \sum_{j=1}^{m}\frac{-q_j z_j^k}{k}. \tag{4.18}$$

For any $p \in \mathbb{N}$ the following estimate holds:

$$\left|\Phi_{E_2}(z) - Q\log z - \sum_{k=1}^{p}\frac{\widetilde{a}_k}{z^k}\right| \leqslant \frac{A}{1-|r/z|}\left|\frac{r}{z}\right|^{p+1} \quad \textit{with} \quad A = \sum_{i=1}^{m}|q_i|. \tag{4.19}$$

The proof proceeds on the lines of the proof of Theorem 4.2.

Statements, allowing us to manipulate the multipole expansion from Theorems 4.2 and 4.3 in a manner required by the fast multipole algorithm, are exactly the same as in [13] and [14]. Thus, we only present them here without giving the corresponding proofs. The following theorem provides a mechanism for shifting the center of a multipole expansion.

Theorem 4.4. *Let $m \in \mathbb{N}$, $r, q_j \in \mathbb{R}$, $z_0 \in \mathbb{C}$ be given, and let for any $z \in \mathbb{C}$ with $|z - z_0| < r$*

$$\widetilde{\Phi}_{H_1,E_1}(z) := \sum_{k=0}^{\infty}\frac{a_k}{(z-z_0)^{k+1}} \quad \textit{with } a_k \textit{ from } (4.14), \tag{4.20}$$

$$\widetilde{\Phi}_{E_2}(z) := \widetilde{a}_0\log(z-z_0) + \sum_{k=1}^{\infty}\frac{\widetilde{a}_k}{(z-z_0)^k} \quad \textit{with } \widetilde{a}_k \textit{ from } (4.18), \tag{4.21}$$

$$\widetilde{\Phi}_{H_2}(z) := \sum_{k=1}^{\infty}\frac{a'_k}{(z-z_0)^{k+1}} \quad \textit{with } a'_k \textit{ from } (4.11). \tag{4.22}$$

Then for any $z \in \mathbb{C}$ with $|z| > R := r + |z_0|$

$$\widetilde{\Phi}_{H_1,E_1}(z) = \sum_{l=1}^{\infty}\frac{b_l}{z^l} \quad \textit{with} \quad b_l = \sum_{k=1}^{l} a_k z_0^{l-k}\binom{l-1}{k-1}, \tag{4.23}$$

$$\widetilde{\Phi}_{E_2}(z) = \widetilde{a}_0\log z + \sum_{l=1}^{\infty}\frac{\widetilde{b}_l}{z^l} \quad \textit{with} \quad \widetilde{b}_l = \sum_{k=1}^{l}\widetilde{a}_k z_0^{l-k}\binom{l-1}{k-1} - \frac{\widetilde{a}_0 z_0^l}{l}, \tag{4.24}$$

$$\widetilde{\Phi}_{H_2}(z) = \sum_{l=1}^{\infty}\frac{b'_l}{z^{l+1}} \quad \textit{with} \quad b'_l = \sum_{k=1}^{l} a'_k z_0^{l-k}\binom{l}{k} \tag{4.25}$$

respectively, and for any $p \in \mathbb{N}$

$$\left|\widetilde{\Phi}_{H_1,E_1}(z) - \sum_{l=1}^{p}\frac{b_l}{z^l}\right| \leqslant \left(A \Big/ \left(1 - \left|\frac{R}{z}\right|\right)\right)\left|\frac{R}{z}\right|^{p+1}, \tag{4.26}$$

$$\left|\widetilde{\Phi}_{E_2}(z) - \widetilde{a}_0\log z - \sum_{l=1}^{p}\frac{\widetilde{b}_l}{z^l}\right| \leqslant \left(A \Big/ \left(1 - \left|\frac{R}{z}\right|\right)\right)\left|\frac{R}{z}\right|^{p+1} \tag{4.27}$$

with $A = \sum_{i=1}^{m} |q_j|$, *and*

$$\left| \widetilde{\Phi}_{H_2}(z) - \sum_{l=1}^{p} \frac{b'_l}{z^{l+1}} \right| \leqslant \alpha \left| \frac{R}{z} \right|^{p+1} \tag{4.28}$$

with

$$\alpha = \frac{A\,(R + (p+1)(|z| - R))}{|z|\,(|z| - R)^2}, \quad i = 1, 2, \quad A = \sum_{i=1}^{m} |q_j \operatorname{Re} z_j|. \tag{4.29}$$

The next theorem describes how to convert the shifted expansion into a local (Taylor) expansion in a circular region of analyticity.

Theorem 4.5. *Let* $m \in \mathbb{N}$, $c, r, q_j \in \mathbb{R}$, *and* $z_0 \in \mathbb{C}$, $j = 1, \ldots, m$, *be given with* $|z_0| > (c+1)r$, *and let* $|z - z_0| < r$ *for all* $z \in \mathbb{C}$. *Then the multipole expansions* (4.20)–(4.22) *converge in the interior of the circle of radius* r *with center at the origin and can be represented by*

$$\widetilde{\Phi}_{H_1,E_1}(z) = \sum_{l=0}^{\infty} b_l\, z^l \quad \text{with} \quad b_l = \frac{1}{z_0^l} \sum_{k=1}^{\infty} \binom{l+k-1}{k-1} \frac{a_k}{(-z_0)^k}, \quad a_k \text{ from } (4.14), \tag{4.30}$$

$$\begin{aligned} \widetilde{\Phi}_{E_2}(z) = \sum_{l=0}^{\infty} \widetilde{b}_l z^l \quad \text{with} \quad \widetilde{b}_0 &= \sum_{k=1}^{\infty} \frac{\widetilde{a}_k}{(-z_0)^k} + \widetilde{a}_0 \log(-z_0), \\ \widetilde{b}_l &= \frac{1}{z_0^l} \sum_{k=1}^{\infty} \binom{l+k-1}{k-1} \frac{\widetilde{a}_k}{(-z_0)^k} - \frac{\widetilde{a}_0}{l z_0^l} \quad \text{for } l \geqslant 1, \quad \widetilde{a}_k \text{ from } (4.18), \end{aligned} \tag{4.31}$$

and

$$\widetilde{\Phi}_{H_2}(z) = \sum_{l=0}^{\infty} b'_l\, z^l \quad \text{with} \quad b'_l = \frac{1}{z_0^{l+1}} \sum_{k=1}^{\infty} (-1) \binom{l+k}{k} \frac{a'_k}{(-z_0)^k}, \quad a'_k \text{ from } (4.11). \tag{4.32}$$

Furthermore, for any $p \in \mathbb{N}$, $p \geqslant \max(2, 2c/(c-1))$, *and the Euler constant* e *the following error bounds for the truncated series hold*:

$$\left| \widetilde{\Phi}_{H_1,E_1}(z) - - \sum_{l=0}^{p} b_l\, z^l \right| \leqslant \frac{4\,A\,ep\,(c+1)}{rc(c-1)} \left(\frac{1}{c} \right)^{p+1} \quad \text{with} \quad A = \sum_{i=1}^{m} |q_j|, \tag{4.33}$$

$$\left| \widetilde{\Phi}_{E_2}(z) - \sum_{l=0}^{p} \widetilde{b}_l z^l \right| \leqslant \frac{A\left(4ep\,(c+1) + c^2\right)}{c\,(c-1)} \left(\frac{1}{c} \right)^{p+1} \quad \text{with} \quad A = \sum_{i=1}^{m} |q_j|, \tag{4.34}$$

$$\left| \widetilde{\Phi}_{H_2}(z) - \sum_{l=0}^{p} b'_l z^l \right| \leqslant \frac{4Aep^2(c+1)}{r^2(p+c-1)^2(c-1)} \left(\frac{1}{c} \right)^{p+1} \quad \text{with} \quad A = \sum_{i=1}^{m} |q_j \operatorname{Re} z_j|. \tag{4.35}$$

The proof of (4.31) and (4.34), can be found, for example, in [8]. The proof of the other statements proceeds analog and was shown, for example, in [13].

The following lemma describes a translation operation for the local (Taylor) expansion. The operation is exact with a finite number of terms. Therefore, no error bound is required.

Lemma 4.6. *For any $z, z_0, a_k \in \mathbb{C}$, $k = 0, 1, \ldots, n$,*

$$\sum_{k=0}^{n} a_k (z - z_0)^k = \sum_{l=0}^{n} \left(\sum_{k=l}^{n} a_k \binom{k}{l} (-z_0)^{k-l} \right) z^l. \tag{4.36}$$

Remark 4.7. A generalization of the presented results to the Stokes equations in three dimensions is the object of the forthcoming work. In such a case, the multipole expansion can be obtained in the terms of spherical harmonics (cf. [15]).

5 Numerical Results

For testing purposes, we consider a domain G with the boundary defined by

$$\begin{aligned} x(t) &:= \cos(2\pi t), \\ y(t) &:= \begin{cases} \sin(2\pi t), & t \in [0, 1/8] \cup [3/8, 1], \\ -\frac{\sqrt{2}}{4}(\cos^4(2\pi t) + cos^2(2\pi t) - 11/4), & t \in]1/8, 3/8[, \end{cases} \end{aligned} \tag{5.1}$$

$t \in [0, 1]$, which is discretized by m discretization points $(x(t_i), y(t_i))$ with $t_i := i/m$, $i = 1, \ldots, m$. We set the density function $\mathbf{q} = 1$ at all the points and computed the velocity part of the hydrodynamical single layer potential (2.9) and its normal stress operator (2.12).

At first, we are interested in the connection between the number p of terms in the expansions and the maximal relative error in the FMM approximation of a hydrodynamical single layer potential $E\,\mathbf{q}$ and its normal stress $H\,\mathbf{q}$ obtained at any of the discretization points. This error is defined via the formulas

$$\begin{aligned} \varepsilon_{\text{rel}}^{E} &:= \max_{i=1,\ldots,n} \left| \frac{\Phi_{dir}^{E}(z_i) - \Phi_{\text{FMM}}^{E}(z_i)}{\Phi_{dir}^{E}(z_i)} \right|, \\ \varepsilon_{\text{rel}}^{H} &:= \max_{i=1,\ldots,n} \left| \frac{\Phi_{dir}^{H}(z_i) - \Phi_{\text{FMM}}^{H}(z_i)}{\Phi_{dir}^{H}(z_i)} \right|, \end{aligned} \tag{5.2}$$

where $\Phi_{dir}^{E}(z_i)$ and $\Phi_{dir}^{H}(z_i)$ denote the hydrodynamical single layer potential and its normal stress obtained by the direct calculation (in double precision), $\Phi_{\text{FMM}}^{E}(z_i)$ and $\Phi_{\text{FMM}}^{H}(z_i)$ - a hydrodynamical single layer potential and its normal stress computed using the fast multipole method. The corresponding results for $m = 1000$ are presented in Table 1.

We compare now the speed and accuracy of the calculation using fast multipole method to those of the direct method. We perform the numerical calculation in two ways in double precision arithmetic: directly, for example, via the direct matrix-vector multiplication, and via the fast multipole method of Greengard and Rokhlin. The number of terms in the expansions p was set to 17 in order to guarantee roughly 6-digit accuracy of the results. The results for the CPU time (in seconds) and the storage (in MB) costs corresponding to the calculation of $E\,\mathbf{q}$ are summarized in Table 2, and corresponding to the calculation of $H\,\mathbf{q}$ are in Table 3. The first column of the tables contains the number m of discretization points, in the second column, l denotes the number of refinements, the third and forth columns present the CPU time in seconds for the direct matrix-vector multiplication t_{dir} and for the computation of matrix-vector product

Table 1: Maximal relative error in the FMM approximation of $E\mathbf{q}$ and $H\mathbf{q}$

p	$\varepsilon_{\text{rel}}^{E}$	$\varepsilon_{\text{rel}}^{H}$
5	$1.23 \cdot 10^{-3}$	$7.26 \cdot 10^{-3}$
10	$6.12 \cdot 10^{-6}$	$8.49 \cdot 10^{-5}$
15	$1.48 \cdot 10^{-7}$	$3.32 \cdot 10^{-6}$
16	$4.28 \cdot 10^{-8}$	$1.13 \cdot 10^{-6}$
17	$2.51 \cdot 10^{-8}$	$6.85 \cdot 10^{-7}$
20	$1.15 \cdot 10^{-9}$	$4.32 \cdot 10^{-8}$

Table 2: Timing and storage results for the calculation of $E\,\mathbf{q}$

m	l	t_{dir}^{E}	t_{FMM}^{E}	additional storage
4000	5	5.01	0.84	0.28
8000	6	20.02	1.82	0.04
16000	7	88.19	3.96	-2.00
32000	8	400.17	8.40	-12.24
64000	9	1984.21	17.98	-57.60

Table 3: Timing and storage results for the calculation of $H\,\mathbf{q}$

m	l	t_{dir}^{H}	t_{FMM}^{H}	additional storage
4000	5	4.05	1.90	0.31
8000	6	17.56	4.09	0.15
16000	7	83.19	8.24	-1.41
32000	8	381.21	17.77	-10.00
64000	9	1784.23	38.08	-49.33

via FMM t_{FMM} respectively. In the last column, we present the information about the additional storage that was needed for the fast multipole method in compare to the direct method.

The numerical experiments show that in the case under consideration, the fast multipole method was more efficient than a direct calculation. From Tables 2 and 3 one can observe that, in the case of the fast multipole method, the CPU time grows almost linearly with the number of discretization points. Moreover, for a number of discretization points more above 16000, the savings in memory are also significant.

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